BACKWARD VOLUME CONTRACTION FOR ENDOMORPHISMS WITH EVENTUAL VOLUME EXPANSION

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Abstract. We consider smooth maps on compact Riemannian manifolds. We prove that under some mild condition of eventual volume expansion Lebesgue almost everywhere we have uniform backward volume contraction on every pre-orbit for Lebesgue almost every point.

1. Statement of results

Let M be a compact Riemannian manifold and let Leb be a volume form on M that we call Lebesgue measure. We take $f\colon M \to M$ any smooth map. Let $0 < a_1 \le a_2 \le a_3 \le \dots$ be a sequence converging to infinity. We define

$$h(x) = \min\{n > 0\colon |\det Df^n(x)| \ge a_n\}, \tag{1}$$

if this minimum exists, and $h(x) = \infty$, otherwise. For $n \ge 1$, we take

$$\Gamma_n = \{x \in M\colon h(x) \ge n\}. \tag{2}$$

Theorem 1.1. *Assume that $h \in L^p(\mathrm{Leb})$, for some $p > 3$, and take $\gamma < (p-3)/(p-1)$. Choose any sequence $0 < b_1 \le b_2 \le b_3 \le \dots$ such that $b_k b_n \ge b_{k+n}$ for every $k, n \in \mathbb{N}$, and assume that there is $n_0 \in \mathbb{N}$ such that $b_n \le \min\{a_n, \mathrm{Leb}(\Gamma_n)^{-\gamma}\}$ for every $n \ge n_0$. Then, for Leb almost every $x \in M$, there exists $C_x > 0$ such that $|\det Df^n(y)| > C_x b_n$ for every $y \in f^{-n}(x)$.*

We say that $f\colon M \to M$ is *eventually volume expanding* if there exists $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\sup_{n\ge 1} \frac{1}{n} \log|\det Df^n(x)| > \lambda. \tag{3}$$

Let h and Γ_n be defined as in (1) and (2), associated to the sequence $a_n = e^{\lambda n}$.

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Corollary 1.2. *If f is eventually volume expanding, then for Lebesgue almost every point $x \in M$ there are $C_x > 0$ and $\sigma_n \to \infty$ such that $|\det Df^n(y)| > C_x \sigma_n$ for every $y \in f^{-n}(x)$. Moreover, given $\alpha > 0$ there is $\beta > 0$ such that*

(1) *if $\mathrm{Leb}(\Gamma_n) \le \mathcal{O}(e^{-\alpha n})$, then we may take $\sigma_n \ge e^{\beta n}$;*
(2) *if $\mathrm{Leb}(\Gamma_n) \le \mathcal{O}(e^{-\alpha n^\tau})$ for some $\tau > 0$, then we may take $\sigma_n \ge e^{\beta n^\tau}$;*
(3) *if $\mathrm{Leb}(\Gamma_n) \le \mathcal{O}(n^{-\alpha})$ and $\alpha > 2$, then we may take $\sigma_n \ge n^\beta$.*

Specific rates will be obtained in Section 4 for some eventually volume expanding endomorphisms. In particular, non-uniformly expanding maps such as quadratic maps and Viana maps will be considered.

2. Concatenated collections

Let $(U_n)_n$ be a collection of measurable subsets of M whose union covers a full Lebesgue measure subset of M. We say that $(U_n)_n$ is a *concatenated collection* if:

$$x \in U_n \quad \text{and} \quad f^n(x) \in U_m \quad \Rightarrow \quad x \in U_{n+m}.$$

Given $x \in \bigcup_{n\ge1} U_n$, we define $u(x)$ as the minimum $n \in \mathbb{N}$ for which $x \in U_n$. Note that by definition we have $x \in U_{u(x)}$. We define the *chain generated by* $x \in \bigcup_{n\ge1} U_n$ as $C(x) = \{x, f(x), \dots, f^{u(x)-1}(x)\}$.

Lemma 2.1. *Let $(U_n)_n$ be a concatenated collection. If*

$$\sum_{n\ge1}\sum_{j=0}^{n-1} \mathrm{Leb}(f^j(u^{-1}(n))) < \infty,$$

then we have $\sup\left\{\, u(y) \,:\, y \in \bigcup_{n\ge1} U_n \text{ and } x \in C(y) \,\right\} < \infty$ *for Lebesgue almost every $x \in M$.*

Proof. Assume that for a given $x \in M$ there exists an infinite number of chains $C_j = \{y_j, f(y_j), \dots, f^{s_j-1}(y_j)\}$, $j \ge 1$, containing x with $s_j \to \infty$. For each $j \ge 1$ let $1 \le r_j < s_j$ be such that $x = f^{r_j}(y_j)$. First we verify that $\lim r_j = \infty$. If not, then replacing by a subsequence, we may assume that there is $N > 0$ such that $r_j < N$ for every $j \ge 1$. This implies that $y_j \in \bigcup_{i=1}^N f^{-i}(x)$ for every $j \ge 1$. Since $\#(\bigcup_{i=1}^N f^{-i}(x)) < \infty$ and the number of chains is infinite, we have a contradiction. Since $r_j \to \infty$ and $x = f^{r_j}(y_j) \in f^{r_j}(u^{-1}(s_j))$, then we have $x \in \bigcup_{n\ge k}\bigcup_{j=0}^{n-1} f^j(u^{-1}(n))$ for every $k \ge 1$. Since we are assuming $\sum_{n\ge1}\sum_{j=0}^{n-1}\mathrm{Leb}(f^j(u^{-1}(n))) < \infty$, we have $\mathrm{Leb}\left(\bigcup_{n\ge k}\bigcup_{j=0}^{n-1} f^j(u^{-1}(n))\right) \to 0$, when $k \to \infty$. This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Let $(U_n)_n$ be a concatenated collection. If*

$$\sup\{\, u(y) \,:\, y \in \cup_{n\geq 1} U_n \text{ and } x \in C(y) \,\} \leq N,$$

then $f^{-n}(x) \subset U_n \cup \cdots \cup U_{n+N}$ for all $n \geq 1$.

Proof. Assume that $\sup\{\, u(y) \,:\, y \in \cup_{n\geq 1} U_n$ and $x \in C(y) \,\} \leq N$, and take $z \in f^{-n}(x)$. Let $z_j = f^j(z)$ for each $j \geq 0$. We distinguish the cases $x \in C(z)$ and $x \notin C(z)$. If $x \in C(z)$, then $n \leq u(z) \leq n+N$. Hence $z \in U_{u(z)} \subset U_n \cup \cdots \cup U_{n+N}$. If $x \notin C(z)$, then letting $u_0 = u(z)$ we must have $u_0 < n$. Let $u_1 = u(z_{u_0})$. If $u_0 + u_1 < n$ we take $u_2 = u(z_{u_0+u_1})$. We proceed in this way until we find the first $s \leq n$ such that $n \leq u_0 + \cdots + u_s$. Note that $u_s = u(z_{u_0+\cdots+u_{s-1}})$, and by the choice of s we must have $x \in C(z_{u_0+\cdots+u_{s-1}})$. Our assumption implies that $u(z_{u_0+\cdots+u_{s-1}}) \leq N$, and so $u_0 + \cdots + u_s \leq n+N$. By construction we have

$$\begin{aligned} z &\in U_{u_0} \\ f^{u_0}(z) = z_{u_0} &\in U_{u_1} \\ f^{u_0+u_1}(z) = z_{u_0+u_1} &\in U_{u_2} \\ &\vdots \\ f^{u_0+\cdots u_{s-1}}(z) = z_{u_0+\cdots u_{s-1}} &\in U_{u_s} \end{aligned}$$

By the definition of a concatenated collection we conclude that $z \in U_{u_0+u_1+\cdots+u_s}$. □

3. Proofs of main results

Let us now prove Theorem 1.2. Suppose that $h \in L^p(\mathrm{Leb})$, for some $p > 3$. This implies that $\sum_{n\geq 1} n^p \,\mathrm{Leb}(h^{-1}(n)) < \infty$, and so there exists some constant $K > 0$ such that

$$\mathrm{Leb}(h^{-1}(n)) \leq K n^{-p}, \quad \text{for every } n \geq 1.$$

Now, taking $0 < \gamma < (p-3)/(p-1)$ we have for some $K' > 0$

$$\sum_{n=1}^{\infty} n \left(\sum_{k=n}^{\infty} \mathrm{Leb}(h^{-1}(k)) \right)^{1-\gamma} \leq \sum_{n=1}^{\infty} n (K'/n^{p-1})^{1-\gamma} < \infty.$$

Defining

$$U_n = \{x \in M \,:\, |\det Df^n(x)| \geq b_n\},$$

then we have that $(U_n)_n$ is a concatenated collection with respect to the Lebesgue measure. Moreover, setting

$$U_n^* = U_n \setminus (U_1 \cup \ldots \cup U_{n-1})$$

one has $U_n^* \subset \bigcup_{m\geq n} h^{-1}(m)$, for otherwise there would be $x \in U_n^* \cap h^{-1}(m)$ with $m < n$, and so $a_m \geq b_m > |\det Df^m(x)| \geq a_m$, which is

not possible. As $|\det Df^j(x)| < b_j$ for every $x \in U_n^*$ and $j < n$, we get $\mathrm{Leb}(f^j(U_n^*)) \le b_j \,\mathrm{Leb}(U_n^*)$ for each $j < n$. Hence

$$\begin{aligned}\sum_{n=n_0+1}^{\infty}\sum_{j=0}^{n-1}\mathrm{Leb}(f^j(U_n^*)) &\le \sum_{n=n_0+1}^{\infty}\sum_{j=0}^{n-1} b_j\,\mathrm{Leb}(U_n^*)\\ &\le \sum_{n=n_0+1}^{\infty}\sum_{j=0}^{n_0-1} b_j\,\mathrm{Leb}(U_n^*) + \sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j\,\mathrm{Leb}(U_n^*)\\ &\le \sum_{j=0}^{n_0-1} b_j + \sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j\,\mathrm{Leb}(U_n^*)\end{aligned}$$

Now we just have to check that the last term in the sum above is finite. Indeed,

$$\begin{aligned}\sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j\,\mathrm{Leb}(U_n^*) &\le \sum_{n=n_0+1}^{\infty}\sum_{j=n_0}^{n-1} b_j \sum_{k=n}^{\infty}\mathrm{Leb}(h^{-1}(k))\\ &\le \sum_{n=n_0+1}^{\infty} n b_n \sum_{k=n}^{\infty}\mathrm{Leb}(h^{-1}(k))\\ &\le \sum_{n=n_0+1}^{\infty} n \left(\sum_{k=n}^{\infty}\mathrm{Leb}(h^{-1}(k)\right)^{-\gamma}\sum_{k=n}^{\infty}\mathrm{Leb}(h^{-1}(k))\\ &= \sum_{n=n_0+1}^{\infty} n \left(\sum_{k=n}^{\infty}\mathrm{Leb}(h^{-1}(k))\right)^{1-\gamma} < \infty.\end{aligned}$$

Applying Lemmas 2.1 and 2.2, we get for each generic point $x \in M$ a positive integer number N_x such that if $y \in f^{-n}(x)$ then $y \in U_{n+s}$ for some $0 \le s \le N_x$. Therefore, $|\det Df^{n+s}(y)| > b_{n+s} \ge b_n$. Then, taking $C_x = K^{-N_x}$, where $K = \sup\{|\det Df(z)|\colon z \in M\}$, we obtain the conclusion of Theorem 1.1:

$$|\det Df^n(y)| = \frac{|\det Df^{n+s}(y)|}{|\det Df^s(x)|} > C_x b_n.$$

Now we explain how we use Theorem 1.1 to prove Corollary 1.2. Recall that in Corollary 1.2 we have $a_n = e^{\lambda n}$ for each $n \in \mathbb{N}$. Assume first that $\mathrm{Leb}(\Gamma_n) \le \mathcal{O}(e^{-c'n})$ for some $c' > 0$. Then it is possible to choose $c > 0$ such that $b_n = e^{cn}$, for $n \ge n_0$. The other two cases are obtained under similar considerations.

4. Examples: non-uniformly expanding maps

An important class of dynamical systems where we can immediately apply our results are the non-uniformly expanding dynamical maps introduced in [2]. As particular examples of this kind of systems we present below quadratic maps and the higher dimensional Viana maps.
Quadratic maps. Let $f_a\colon [-1,1] \to [-1,1]$ be given by $f_a(x) = 1 - ax^2$, for $0 < a \leq 2$. Results in [3, 8] give that for a positive Lebesgue measure set of parameters f_a in non-uniformly expanding. Ongoing work [5] gives that for a positive Lebesgue measure set of parameters there are $C, c > 0$ such that $\mathrm{Leb}(\Gamma_n) \leq Ce^{-cn}$ for every $n \geq 1$.

Thus, it follows from Corollary 1.2 that *we may find $\beta > 0$ such for Lebesgue almost every $x \in I$ there is $C_x > 0$ such that $|(f^n)'(y)| > C_x e^{\beta n}$ for every $y \in f^{-n}(x)$.*
Viana maps. Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ given by $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map $\hat{f}$ from $S^1 \times \mathbb{R}$ into itself given by $\hat{f}(s,x) = \big(\hat{g}(s), \hat{q}(s,x)\big)$, where $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, and $\hat{g}$ is the uniformly expanding map of S^1 defined by $\hat{g}(s) = ds \pmod{\mathbb{Z}}$ for some integer $d \geq 2$. For $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. Moreover, there are $C, c > 0$ such that $\mathrm{Leb}(\Gamma_n) \leq Ce^{-c\sqrt{n}}$ for every $n \geq 1$; see [1, 4, 9].

Thus, it follows from Corollary 1.2 that *we may find $\beta > 0$ such for Lebesgue almost every $X \in S^1 \times I$ there is a constant $C_X > 0$ such that $|\det Df^n(Y)| > C_X e^{\beta\sqrt{n}}$ for every $Y \in f^{-n}(X)$.*

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